

# On a certain subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$ related to the degenerate $q$ -Onsager algebra

Tomoya Hattai and Tatsuro Ito

Division of Mathematical and Physical Sciences, Kanazawa University,  
Kakuma-machi, Kanazawa 920-1192, Japan

## Abstract

In [4], it is discussed that a certain subalgebra of the quantum affine algebra  $U_q(\widehat{\mathfrak{sl}}_2)$  controls the type I TD-algebra of the second kind (the degenerate  $q$ -Onsager algebra). The subalgebra, which we denote by  $U'_q(\widehat{\mathfrak{sl}}_2)$ , is generated by  $e_0^+$ ,  $e_1^+$ ,  $k_i^{\pm 1}$  ( $i = 0, 1$ ),  $e_0^-$  missing from the Chevalley generators  $e_i^{\pm}$ ,  $k_i^{\pm 1}$  ( $i = 0, 1$ ) of  $U_q(\widehat{\mathfrak{sl}}_2)$ . In this paper, we determine the finite-dimensional irreducible representations of  $U'_q(\widehat{\mathfrak{sl}}_2)$ . Intertwiners are also determined.

**Keywords.** Degenerate  $q$ -Onsager algebra, quantum affine algebra, TD-algebra, augmented TD-algebra, TD-pair, Terwilliger algebra, P- and Q-polynomial association scheme.

**2010 Mathematics Subject Classification.** Primary: 17B37. Secondary: 05E30.

## 1 Introduction

Throughout this paper, the ground field is  $\mathbb{C}$  and  $q$  stands for a nonzero scalar that is not a root of unity. The symbols  $\varepsilon, \varepsilon^*$  stand for an integer chosen from  $\{0, 1\}$ . Let  $\mathcal{A}_q = \mathcal{A}_q^{(\varepsilon, \varepsilon^*)}$  denote the associative algebra with 1 generated by  $z, z^*$  subject to the defining relations [4]

$$(TD) \quad \begin{cases} [z, [z, [z, z^*]_q]_{q^{-1}} = -\varepsilon(q^2 - q^{-2})^2 [z, z^*], \\ [z^*, [z^*, [z^*, z]_q]_{q^{-1}} = -\varepsilon^*(q^2 - q^{-2})^2 [z^*, z], \end{cases} \quad (1)$$

where  $[X, Y] = XY - YX$ ,  $[X, Y]_q = qXY - q^{-1}YX$ . This paper deals with a subalgebra of the quantum affine algebra  $U_q(\widehat{\mathfrak{sl}}_2)$  that is closely related to  $\mathcal{A}_q$  in the case of  $(\varepsilon, \varepsilon^*) = (1, 0)$ . If  $(\varepsilon, \varepsilon^*) = (0, 0)$ ,  $\mathcal{A}_q$  is isomorphic to the positive part of  $U_q(\widehat{\mathfrak{sl}}_2)$ . If  $(\varepsilon, \varepsilon^*) = (1, 1)$ ,  $\mathcal{A}_q$  is called the  $q$ -Onsager algebra. If  $(\varepsilon, \varepsilon^*) = (1, 0)$ ,  $\mathcal{A}_q$  may well be called the degenerate  $q$ -Onsager algebra.

The algebra  $\mathcal{A}_q$  arises in the course of the classification of TD-pairs of type I, which is a critically important step in the study of representations of Terwilliger algebras for P- and Q- polynomial association schemes [3]. For this reason,  $\mathcal{A}_q$  is called the TD-algebra of type I. Precisely speaking, the TD-algebra of type I is standardized to be the algebra  $\mathcal{A}_q$ , where  $q$  is the main parameter for TD-pairs of type I; so  $q \neq \pm 1$  and  $q$  is allowed to be a root of unity. In our case where we assume  $q$  is not a root of unity, to classify the TD-pairs of type I is to determine the finite-dimensional irreducible representations  $\rho : \mathcal{A}_q \longrightarrow \text{End}(V)$  with the property that  $\rho(z)$ ,  $\rho(z^*)$  are both diagonalizable, and vice versa. Such irreducible representations of  $\mathcal{A}_q$  are determined in [4] via embeddings of  $\mathcal{A}_q$  into the augmented TD-algebra  $\mathcal{T}_q$ . (In the case of  $(\varepsilon, \varepsilon^*) = (1, 1)$ , the diagonalizability condition of  $\rho(z)$ ,  $\rho(z^*)$  can be dropped, because it turns out that this condition always holds for every finite-dimensional irreducible representation  $\rho$  of the  $q$ -Onsager algebra  $\mathcal{A}_q$ .)  $\mathcal{T}_q$  is easier than  $\mathcal{A}_q$  to study representations for, and each finite-dimensional irreducible representation  $\rho : \mathcal{A}_q \longrightarrow \text{End}(V)$  with  $\rho(z)$ ,  $\rho(z^*)$  diagonalizable can be extended to a finite-dimensional irreducible representation of  $\mathcal{T}_q$  via a certain embedding of  $\mathcal{A}_q$  into  $\mathcal{T}_q$ .

The augmented TD-algebra  $\mathcal{T}_q = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)}$  is the associative algebra with 1 generated by  $x$ ,  $y$ ,  $k^{\pm 1}$  subject to the defining relations

$$(\text{TD})_0 \begin{cases} kk^{-1} = k^{-1}k = 1, \\ kxk^{-1} = q^2x, \\ kyk^{-1} = q^{-2}y, \end{cases} \quad (2)$$

and

$$(\text{TD})_1 \begin{cases} [x, [x, [x, y]_q]_{q^{-1}}] = \delta(\varepsilon^*x^2k^2 - \varepsilon k^{-2}x^2), \\ [y, [y, [y, x]_q]_{q^{-1}}] = \delta(-\varepsilon^*k^2y^2 + \varepsilon y^2k^{-2}), \end{cases} \quad (3)$$

where  $\delta = -(q - q^{-1})(q^2 - q^{-2})(q^3 - q^{-3})q^4$ . The finite-dimensional irreducible representations of  $\mathcal{T}_q$  are determined in [4] via embeddings of  $\mathcal{T}_q$  into the  $U_q(\mathfrak{sl}_2)$ -loop algebra  $U_q(L(\mathfrak{sl}_2))$ .

Let  $e_i^\pm, k_i^{\pm 1}$  ( $i = 0, 1$ ) be the Chevalley generators of  $U_q(L(\mathfrak{sl}_2))$ . So the defining relations of  $U_q(L(\mathfrak{sl}_2))$  are

$$\begin{cases} k_0 k_1 = k_1 k_0 = 1, \\ k_i k_i^{-1} = k_i^{-1} k_i = 1, \\ k_i e_i^\pm k_i^{-1} = q^{\pm 2} e_i^\pm, \\ k_i e_j^\pm k_i^{-1} = q^{\mp 2} e_j^\pm \quad (i \neq j), \\ [e_i^+, e_i^-] = \frac{k_i - k_i^{-1}}{q - q^{-1}}, \\ [e_i^+, e_j^-] = 0 \quad (i \neq j), \\ [e_i^\pm, [e_i^\pm, [e_i^\pm, e_j^\pm]_q]_{q^{-1}}] = 0 \quad (i \neq j). \end{cases} \quad (4)$$

Note that if  $k_0 k_1 = k_1 k_0 = 1$  is replaced by  $k_0 k_1 = k_1 k_0$  in (4), we have the quantum affine algebra  $U_q(\widehat{\mathfrak{sl}}_2)$ :  $U_q(L(\mathfrak{sl}_2))$  is the quotient algebra of  $U_q(\widehat{\mathfrak{sl}}_2)$  by the two-sided ideal generated by  $k_0 k_1 - 1$ . For a nonzero scalar  $s$ , define the elements  $x(s), y(s), k(s)$  of  $U_q(L(\mathfrak{sl}_2))$  by

$$\begin{cases} x(s) = -q^{-1}(q - q^{-1})^2 (s e_0^+ + \varepsilon s^{-1} e_1^- k_1), \\ y(s) = \varepsilon^* s e_0^- k_0 + s^{-1} e_1^+, \\ k(s) = s k_0. \end{cases} \quad (5)$$

Then the mapping

$$\varphi_s : \mathcal{T}_q \longrightarrow U_q(L(\mathfrak{sl}_2)) \quad (x, y, k \mapsto x(s), y(s), k(s)) \quad (6)$$

gives an injective algebra homomorphism. If  $(\varepsilon, \varepsilon^*) = (0, 0)$ , the image  $\varphi_s(\mathcal{T}_q)$  coincides with the Borel subalgebra generated by  $e_i^\pm, k_i^{\pm 1}$  ( $i = 0, 1$ ). If  $(\varepsilon, \varepsilon^*) = (1, 0)$ , the image  $\varphi_s(\mathcal{T}_q)$  is properly contained in the subalgebra generated by  $e_0^+, e_1^\pm, k_i^{\pm 1}$  ( $i = 0, 1$ ),  $e_0^-$  missing from the generators; we denote this subalgebra by  $U'_q(L(\mathfrak{sl}_2))$ . Through the natural homomorphism  $U_q(\widehat{\mathfrak{sl}}_2) \longrightarrow U_q(L(\mathfrak{sl}_2))$ , pull back the subalgebra  $U'_q(L(\mathfrak{sl}_2))$  and denote the pre-image by  $U'_q(\widehat{\mathfrak{sl}}_2)$ :

$$U'_q(\widehat{\mathfrak{sl}}_2) = \langle e_0^+, e_1^\pm, k_i^{\pm 1} \mid i = 0, 1 \rangle \subset U_q(\widehat{\mathfrak{sl}}_2). \quad (7)$$

In [4], it is shown that in the case of  $(\varepsilon, \varepsilon^*) = (1, 0)$ , all the finite-dimensional irreducible representations of  $\mathcal{T}_q$  are produced by tensor products of evaluation modules for  $U'_q(L(\mathfrak{sl}_2))$  via the embedding  $\varphi_s$  of  $\mathcal{T}_q$  into  $U'_q(L(\mathfrak{sl}_2))$ . Using this fact and the Drinfel'd polynomials, we show in this paper that there are no finite-dimensional irreducible representations of  $U'_q(L(\mathfrak{sl}_2))$

and hence of  $U'_q(\widehat{\mathfrak{sl}}_2)$  other than those afforded by tensor products of evaluation modules, if we apply suitable automorphisms of  $U'_q(L(\mathfrak{sl}_2))$ ,  $U'_q(\widehat{\mathfrak{sl}}_2)$  to adjust the types of the representations to be  $(1, 1)$ . Here we note that the evaluation parameters are allowed to be zero for  $U'_q(L(\mathfrak{sl}_2))$ ,  $U'_q(\widehat{\mathfrak{sl}}_2)$ . Details will be discussed in Section 2, where the isomorphism classes of finite-dimensional irreducible representations of  $U'_q(\widehat{\mathfrak{sl}}_2)$  are also determined. In Section 3, intertwiners will be determined for finite-dimensional irreducible  $U'_q(\widehat{\mathfrak{sl}}_2)$ -modules.

Drinfel'd polynomials are not the main subject of this paper but the key tool for the classification of finite-dimensional irreducible representations of  $U_q(\widehat{\mathfrak{sl}}_2)$ ,  $U'_q(\widehat{\mathfrak{sl}}_2)$ . They are defined in [4], directly attached to  $\mathcal{T}_q$ -modules, not to  $U_q(\widehat{\mathfrak{sl}}_2)$ - or  $U'_q(\widehat{\mathfrak{sl}}_2)$ -modules. (In the case of  $(\varepsilon, \varepsilon^*) = (0, 0)$ , they turn out to coincide with the original ones up to the reciprocal of the variable.) So if our approach is applied to the case of  $(\varepsilon, \varepsilon^*) = (0, 0)$ , finite-dimensional irreducible representations are naturally classified in the first place for the Borel subalgebra of  $U_q(\widehat{\mathfrak{sl}}_2)$  and then for  $U_q(\widehat{\mathfrak{sl}}_2)$  itself. This reverses the process adopted in [1] and will be briefly demonstrated in Section 2 as a warm-up for the case of  $(\varepsilon, \varepsilon^*) = (1, 0)$ , thus giving another proof to the classical result of Chari-Pressley [2].

## 2 Finite-dimensional irreducible representations of $U'_q(\widehat{\mathfrak{sl}}_2)$

The subalgebra  $U'_q(\widehat{\mathfrak{sl}}_2)$  of the quantum affine algebra  $U_q(\widehat{\mathfrak{sl}}_2)$  is generated by  $e_0^+$ ,  $e_1^\pm$ ,  $k_i^{\pm 1}$  ( $i = 0, 1$ ),  $e_0^-$  missing from the generators, and has by the triangular decomposition of  $U_q(\widehat{\mathfrak{sl}}_2)$  the defining relations

$$\left\{ \begin{array}{l} k_0 k_1 = k_1 k_0, \\ k_i k_i^{-1} = k_i^{-1} k_i = 1, \\ k_0 e_0^+ k_0^{-1} = q^2 e_0^+, \quad k_1 e_1^\pm k_1^{-1} = q^{\pm 2} e_1^\pm, \\ k_1 e_0^+ k_1^{-1} = q^{-2} e_0^+, \quad k_0 e_1^\pm k_0^{-1} = q^{\mp 2} e_1^\pm, \\ [e_1^+, e_1^-] = \frac{k_1 - k_1^{-1}}{q - q^{-1}}, \\ [e_0^+, e_1^-] = 0, \\ [e_i^+, [e_i^+, [e_i^+, e_j^+]_q]_{q^{-1}}] = 0 \quad (i \neq j). \end{array} \right. \quad (8)$$

Note that if  $k_0k_1 = k_1k_0$  is replaced by  $k_0k_1 = k_1k_0 = 1$  in (8), we have the defining relations for  $U'_q(L(\mathfrak{sl}_2))$ .

Let  $V$  be a finite-dimensional irreducible  $U'_q(\widehat{\mathfrak{sl}}_2)$ -module. Let us first observe that the  $U'_q(\widehat{\mathfrak{sl}}_2)$ -module  $V$  is obtained from a  $U'_q(L(\mathfrak{sl}_2))$ -module by applying an automorphism of  $U'_q(\widehat{\mathfrak{sl}}_2)$ . Since the element  $k_0k_1$  belongs to the centre of  $U'_q(\widehat{\mathfrak{sl}}_2)$ ,  $k_0k_1$  acts on  $V$  as a scalar  $s$  by Schur's lemma. Since  $k_0k_1$  is invertible, the scalar  $s$  is nonzero:  $k_0k_1|_V = s \in \mathbb{C}^\times$ . Observe that  $U'_q(\widehat{\mathfrak{sl}}_2)$  admits an automorphism that sends  $k_0$  to  $s^{-1}k_0$  and preserves  $k_1$ . Hence we may assume  $k_0k_1|_V = 1$ . Then we can regard  $V$  as a  $U'_q(L(\mathfrak{sl}_2))$ -module.

Let  $V$  be a finite-dimensional irreducible  $U'_q(L(\mathfrak{sl}_2))$ -module. For a scalar  $\theta$ , set  $V(\theta) = \{v \in V \mid k_0v = \theta v\}$ . So if  $V(\theta) \neq 0$ ,  $\theta$  is an eigenvalue of  $k_0$  and  $V(\theta)$  is the corresponding eigenspace of  $k_0$ . For an eigenvalue  $\theta$  and an eigenvector  $v \in V(\theta)$ , it holds that  $e_0^\pm v \in V(q^{\pm 2}\theta)$  by the relation  $k_0e_0^\pm = q^2e_0^\pm k_0$  and  $e_1^\pm v \in V(q^{\mp 2}\theta)$  by  $k_0e_1^\pm = q^{\mp 2}e_1^\pm k_0$ . We have

$$e_0^\pm V(\theta) \subseteq V(q^{\pm 2}\theta), \quad e_1^\pm V(\theta) \subseteq V(q^{\mp 2}\theta). \quad (9)$$

If  $\dim V = 1$ , then  $e_0^\pm V = 0$ ,  $e_1^\pm V = 0$  by (9) and  $k_0|_V = \pm 1$  by  $[e_1^+, e_1^-] = (k_1 - k_1^{-1})/(q - q^{-1}) = (k_0^{-1} - k_0)/(q - q^{-1})$ . Such a  $U'_q(L(\mathfrak{sl}_2))$ -module  $V$  is said to be *trivial*. Assume  $\dim V \geq 2$ . Choose an eigenvalue  $\theta$  of  $k_0$  on  $V$ . Then  $\sum_{i \in \mathbb{Z}} V(q^{\pm 2i}\theta)$  is invariant under the actions of the generators by (9), and so we have  $V = \sum_{i \in \mathbb{Z}} V(q^{\pm 2i}\theta)$  by the irreducibility of the  $U'_q(L(\mathfrak{sl}_2))$ -module  $V$ . Since  $V$  is finite-dimensional, there exists a positive integer  $d$  and a nonzero scalar  $s_0$  such that the eigenspace decomposition of  $k_0$  is

$$V = \bigoplus_{i=0}^d V(s_0 q^{2i-d}). \quad (10)$$

We want to show that  $s_0 = \pm 1$  holds in (10).

Consider the subalgebra of  $U'_q(L(\mathfrak{sl}_2))$  generated by  $e_1^\pm$ ,  $k_1^{\pm 1}$  and denote it by  $\mathcal{U}$ :  $\mathcal{U} = \langle e_1^\pm, k_1^{\pm 1} \rangle$ . Regard  $V$  as a  $\mathcal{U}$ -module. Since  $\mathcal{U}$  is isomorphic to the quantum algebra  $U_q(\mathfrak{sl}_2)$ ,  $V$  is a direct sum of irreducible  $\mathcal{U}$ -modules, and for each irreducible  $\mathcal{U}$ -submodule  $W$  of  $V$ , the eigenvalues of  $k_1 = k_0^{-1}$  on  $W$  are either  $\{q^{2i-\ell} \mid 0 \leq i \leq \ell\}$  or  $\{-q^{2i-\ell} \mid 0 \leq i \leq \ell\}$  for some nonnegative integer  $\ell$ . In particular, if  $\theta$  is an eigenvalue of  $k_0$ , so is  $\theta^{-1}$ . The collection of such eigenvalues gives rise to the eigenspace decomposition of (10). We obtain  $s_0 = \pm 1$ . Observe that  $U'_q(L(\mathfrak{sl}_2))$  admits an automorphism that sends  $k_i$  to

$-k_i$  ( $i = 0, 1$ ) and  $e_1^+$  to  $-e_1^+$ . Hence we may assume  $s_0 = 1$  in (10). Note that in this case,  $k_1$  has the eigenvalues  $\{s_1 q^{2i-\ell} \mid 0 \leq i \leq \ell\}$  with  $s_1 = 1$ . Such an irreducible module or the irreducible representation afforded by such is said to be of *type*  $(1, 1)$ , indicating  $(s_0, s_1) = (1, 1)$ . We conclude that the determination of finite-dimensional irreducible representations for  $U'_q(\widehat{\mathfrak{sl}}_2)$  is, via automorphisms, reduced to that of type  $(1, 1)$  for  $U'_q(L(\mathfrak{sl}_2))$ .

In the rest of this section, we shall introduce evaluation modules for  $U'_q(L(\mathfrak{sl}_2))$  and show that every finite-dimensional irreducible representation of type  $(1, 1)$  of  $U'_q(L(\mathfrak{sl}_2))$  is afforded by a tensor product of them. For  $a \in \mathbb{C}$  and  $\ell \in \mathbb{Z}_{\geq 0}$ , let  $V(\ell, a)$  denote the  $(\ell + 1)$ -dimensional vector space with a basis  $v_0, v_1, \dots, v_\ell$ . Using (8), it can be routinely verified that  $U'_q(L(\mathfrak{sl}_2))$  acts on  $V(\ell, a)$  by

$$\begin{cases} k_0 v_i = q^{2i-\ell} v_i, \\ k_1 v_i = q^{\ell-2i} v_i, \\ e_0^+ v_i = a q [i+1] v_{i+1}, \\ e_1^+ v_i = [\ell-i+1] v_{i-1}, \\ e_1^- v_i = [i+1] v_{i+1}, \end{cases} \quad (11)$$

where  $v_{-1} = v_{\ell+1} = 0$  and  $[t] = [t]_q = (q^t - q^{-t})/(q - q^{-1})$ . This  $U'_q(L(\mathfrak{sl}_2))$ -module  $V(\ell, a)$  is irreducible and called an *evaluation module*. The basis  $v_0, v_1, \dots, v_\ell$  of the  $U'_q(L(\mathfrak{sl}_2))$ -module  $V(\ell, a)$  is called a *standard basis*. The vector  $v_0$  is called the *highest weight vector*. Note that the evaluation parameter  $a$  is allowed to be zero. Also note that if  $\ell = 0$ ,  $V(\ell, a)$  is the trivial module. We denote the evaluation module  $V(\ell, 0)$  by  $V(\ell)$ , allowing  $\ell = 0$ , and use the notation  $V(\ell, a)$  only for an evaluation module with  $a \neq 0$  and  $\ell \geq 1$ .

The  $U_q(\mathfrak{sl}_2)$ -loop algebra  $U_q(L(\mathfrak{sl}_2))$  has the coproduct  $\Delta : U_q(L(\mathfrak{sl}_2)) \longrightarrow U_q(L(\mathfrak{sl}_2)) \otimes U_q(L(\mathfrak{sl}_2))$  defined by

$$\begin{cases} \Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}, \\ \Delta(e_i^+) = k_i \otimes e_i^+ + e_i^+ \otimes 1, \\ \Delta(e_i^- k_i) = k_i \otimes e_i^- k_i + e_i^- k_i \otimes 1. \end{cases} \quad (12)$$

The subalgebra  $U'_q(L(\mathfrak{sl}_2))$  is closed under  $\Delta$ . Thus given a set of evaluation modules  $V(\ell), V(\ell_i, a_i)$  ( $1 \leq i \leq n$ ) for  $U'_q(L(\mathfrak{sl}_2))$ , the tensor product

$$V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n) \quad (13)$$

becomes a  $U'_q(L(\mathfrak{sl}_2))$ -module via  $\Delta$ . Note that by the coassociativity of  $\Delta$ , the way of putting parentheses in the tensor product of (13) does not affect the isomorphism class as a  $U'_q(L(\mathfrak{sl}_2))$ -module. Also note that if  $\ell = 0$  in (13), then  $V(0)$  is the trivial module and the tensor product of (13) is isomorphic to  $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$  as  $U'_q(L(\mathfrak{sl}_2))$ -modules. Finally we allow  $n = 0$ , in which case we understand that the tensor product of (13) means  $V(\ell)$ .

With the evaluation module  $V(\ell, a)$ , we associate the set  $S(\ell, a)$  of scalars  $aq^{-\ell+1}, aq^{-\ell+3}, \dots, aq^{\ell-1}$ :

$$S(\ell, a) = \{aq^{2i-\ell+1} \mid 0 \leq i \leq \ell-1\}. \quad (14)$$

The set  $S(\ell, a)$  is called a  $q$ -string of length  $\ell$ . Two  $q$ -strings  $S(\ell, a), S(\ell', a')$  are said to be *in general position* if either

- (i) the union  $S(\ell, a) \cup S(\ell', a')$  is not a  $q$ -string,
- or
- (ii) one of  $S(\ell, a), S(\ell', a')$  includes the other.

Below is the main theorem of this paper. It classifies the isomorphism classes of the finite-dimensional irreducible  $U'_q(L(\mathfrak{sl}_2))$ -modules of type  $(1, 1)$ .

**Theorem 1.** The following (i), (ii), (iii) holds.

- (i) A tensor product  $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$  of evaluation modules is irreducible as a  $U'_q(L(\mathfrak{sl}_2))$ -module if and only if  $S(\ell_i, a_i), S(\ell_j, a_j)$  are in general position for all  $i, j \in \{1, 2, \dots, n\}$ . In this case,  $V$  is of type  $(1, 1)$ .
- (ii) Consider two tensor products  $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ ,  $V' = V(\ell') \otimes V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_m, a'_m)$  of evaluation modules and assume that they are both irreducible as a  $U'_q(L(\mathfrak{sl}_2))$ -module. Then  $V, V'$  are isomorphic as  $U'_q(L(\mathfrak{sl}_2))$ -modules if and only if  $\ell = \ell', n = m$  and  $(\ell_i, a_i) = (\ell'_i, a'_i)$  for all  $i$  ( $1 \leq i \leq n$ ) with a suitable reordering of the evaluation modules  $V(\ell_1, a_1), \dots, V(\ell_n, a_n)$ .
- (iii) Every non-trivial finite-dimensional irreducible  $U'_q(L(\mathfrak{sl}_2))$ -module of type  $(1, 1)$  is isomorphic to some tensor product  $V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$  of evaluation modules.

Discard the evaluation module  $V(\ell)$  from the statement of Theorem 1 and replace  $U'_q(L(\mathfrak{sl}_2))$  by  $U_q(L(\mathfrak{sl}_2))$  or  $\mathcal{B}$ , where  $\mathcal{B}$  is the Borel subalgebra of  $U_q(L(\mathfrak{sl}_2))$  generated by  $e_i^+$ ,  $k_i^{\pm 1}$  ( $i = 0, 1$ ). Then it precisely describes the classification of the isomorphism classes of finite-dimensional irreducible modules of type  $(1, 1)$  for  $U_q(L(\mathfrak{sl}_2))$  [2] or  $\mathcal{B}$  [1]. There is a one-to-one correspondence of finite-dimensional irreducible modules of type  $(1, 1)$  between  $U_q(L(\mathfrak{sl}_2))$  and  $\mathcal{B}$ : every finite-dimensional irreducible  $U_q(L(\mathfrak{sl}_2))$ -module of type  $(1, 1)$  is irreducible as a  $\mathcal{B}$ -module and every finite-dimensional irreducible  $\mathcal{B}$ -module of type  $(1, 1)$  is uniquely extended to a  $U_q(L(\mathfrak{sl}_2))$ -module. This sort of correspondence of finite-dimensional irreducible modules exists between  $U'_q(L(\mathfrak{sl}_2))$  and  $\mathcal{T}_q$  via the embedding  $\varphi_s$  of (6), where  $\mathcal{T}_q$  is the augmented TD-algebra with  $(\varepsilon, \varepsilon^*) = (1, 0)$ , and this gives a proof of Theorem 1. The key to our understanding of the correspondence is the following two lemmas. Let  $\mathcal{U}$  denote the quantum algebra  $U_q(\mathfrak{sl}_2)$ :  $\mathcal{U}$  is the associative algebra with 1 generated by  $X^\pm$ ,  $K^{\pm 1}$  subject to the defining relations

$$\begin{cases} KK^{-1} = K^{-1}K = 1, \\ KX^\pm K^{-1} = q^{\pm 2}X^\pm, \\ [X^+, X^-] = \frac{K - K^{-1}}{q - q^{-1}}. \end{cases} \quad (15)$$

**Lemma 1.** [4, Lemma 7.5] Let  $V$  be a finite-dimensional  $\mathcal{U}$ -module that has the following weight-space ( $K$ -eigenspace) decomposition:

$$V = \bigoplus_{i=0}^d U_i, \quad K|_{U_i} = q^{2i-d} \quad (0 \leq i \leq d).$$

Let  $W$  be a subspace of  $V$  as a vector space. Assume that  $W$  is invariant under the actions of  $X^+$  and  $K$ :

$$X^+W \subseteq W, \quad KW \subseteq W.$$

If it holds that

$$\dim(W \cup U_i) = \dim(W \cup U_{d-i}) \quad (0 \leq i \leq d),$$

then  $X^-W \subseteq W$ , i.e.,  $W$  is a  $\mathcal{U}$ -submodule.

**Lemma 2.** If  $V$  is a finite-dimensional  $\mathcal{U}$ -module, the action of  $X^-$  on  $V$  is uniquely determined by those of  $X^+$ ,  $K^{\pm 1}$  on  $V$ .



*Proof.* The claim holds if  $V$  is irreducible as a  $\mathcal{U}$ -module. By the semi-simplicity of  $\mathcal{U}$ , it holds generally.  $\square$

As a warm-up for the proof of Theorem 1, we shall demonstrate how to use these lemmas in the case of the corresponding theorem [2] for  $U_q(L(\mathfrak{sl}_2))$ . We want, and it is enough, to show part (iii) of the theorem for  $U_q(L(\mathfrak{sl}_2))$  by using the classification of finite-dimensional irreducible  $\mathcal{B}$ -modules, since the parts (i), (ii) are well-known in advance of [2] and the finite-dimensional irreducible  $\mathcal{B}$ -modules are classified in [4] without using the part (iii) in question.

Let  $V$  be a finite-dimensional irreducible  $U_q(L(\mathfrak{sl}_2))$ -module of type  $(1, 1)$ . Then  $V$  has the weight-space decomposition

$$V = \bigoplus_{i=0}^d U_i, \quad k_0|_{U_i} = q^{2i-d} \quad (0 \leq i \leq d).$$

Regard  $V$  as a  $\mathcal{B}$ -module. Let  $W$  be a minimal  $\mathcal{B}$ -submodule of  $V$ . Note that  $W$  is irreducible as a  $\mathcal{B}$ -module. We want to show  $W = V$ , i.e.,  $V$  is irreducible as a  $\mathcal{B}$ -module. Since the mapping  $(e_0^+)^{d-2i} : U_i \rightarrow U_{d-i}$  is a bijection and  $W \cap U_i$  is mapped into  $W \cap U_{d-i}$  by  $(e_0^+)^{d-2i}$ , we have  $\dim(W \cap U_i) \leq \dim(W \cap U_{d-i})$  ( $0 \leq i \leq [d/2]$ ). Similarly from the bijection  $(e_1^+)^{d-2i} : U_{d-i} \rightarrow U_i$ , we get  $\dim(W \cap U_{d-i}) \leq \dim(W \cap U_i)$ . Thus it holds that

$$\dim(W \cap U_i) = \dim(W \cap U_{d-i}) \quad (0 \leq i \leq d).$$

Consider the algebra homomorphism from  $\mathcal{U}$  to  $U_q(L(\mathfrak{sl}_2))$  that sends  $X^+$ ,  $X^-$ ,  $K^{\pm 1}$  to  $e_0^+$ ,  $e_0^-$ ,  $k_0^{\pm 1}$ , respectively. Regard  $V$  as a  $\mathcal{U}$ -module via this algebra homomorphism. Then  $X^+W \subseteq W$ ,  $KW \subseteq W$ . Since  $\dim(W \cap U_i) = \dim(W \cap U_{d-i})$  ( $0 \leq i \leq d$ ), we have by Lemma 1 that  $X^-W \subseteq W$ , i.e.,  $e_0^-W \subseteq W$ . Similarly, Lemma 1 can be applied to the  $\mathcal{U}$ -module  $V$  via the algebra homomorphism from  $\mathcal{U}$  to  $U_q(L(\mathfrak{sl}_2))$  that sends  $X^+$ ,  $X^-$ ,  $K^{\pm 1}$  to  $e_1^+$ ,  $e_1^-$ ,  $k_1^{\pm 1}$ , respectively, in which case the weight-space decomposition of the  $\mathcal{U}$ -module  $V$  is  $V = \bigoplus_{i=0}^d U_{d-i}$ ,  $K|_{U_{d-i}} = q^{2i-d}$  ( $0 \leq i \leq d$ ). Consequently, we get  $X^-W \subseteq W$ , i.e.,  $e_1^-W \subseteq W$ . Thus  $W$  is  $U_q(L(\mathfrak{sl}_2))$ -invariant and we have  $W = V$  by the irreducibility of the  $U_q(L(\mathfrak{sl}_2))$ -module  $V$ . We conclude that every finite-dimensional irreducible  $U_q(L(\mathfrak{sl}_2))$ -module of type  $(1, 1)$  is irreducible as a  $\mathcal{B}$ -module.

Now consider the class of finite-dimensional irreducible  $\mathcal{B}$ -modules  $V$ , where  $V$  runs through all tensor products of evaluation modules that are

irreducible as a  $U_q(L(\mathfrak{sl}_2))$ -module:

$$V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n).$$

Then it turns out that the Drinfel'd polynomials  $P_V(\lambda)$  of the irreducible  $\mathcal{B}$ -modules  $V$  exhaust all that are possible for finite-dimensional irreducible  $\mathcal{B}$ -modules of type  $(1, 1)$ , as shown in [4] by the product formula

$$P_V(\lambda) = \prod_{i=1}^n P_{V(\ell_i, a_i)}(\lambda),$$

$$P_{V(\ell_i, a_i)}(\lambda) = \prod_{\zeta \in S(\ell_i, a_i)} (\lambda + \zeta).$$

Since the Drinfel'd polynomial  $P_V(\lambda)$  determines the isomorphism class of the  $\mathcal{B}$ -module  $V$  of type  $(1, 1)$  [4, the injectivity part of Theorem 1.9'], there are no other finite-dimensional irreducible  $\mathcal{B}$ -modules of type  $(1, 1)$ . This means that every finite-dimensional irreducible  $\mathcal{B}$ -module of type  $(1, 1)$  comes from some tensor product of evaluation modules for  $U_q(L(\mathfrak{sl}_2))$ .

Let  $V$  be a finite-dimensional irreducible  $U_q(L(\mathfrak{sl}_2))$ -module of type  $(1, 1)$ . Then  $V$  is irreducible as a  $\mathcal{B}$ -module and so there exists an irreducible  $U_q(L(\mathfrak{sl}_2))$ -module  $V' = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$  such that  $V, V'$  are isomorphic as  $\mathcal{B}$ -modules. By Lemma 2,  $V, V'$  are isomorphic as  $U_q(L(\mathfrak{sl}_2))$ -modules. This completes the proof of part (iii) of the theorem for  $U_q(L(\mathfrak{sl}_2))$ .

The proof of Theorem 1 can be given by an argument very similar to the one we have seen above for the case of  $U_q(L(\mathfrak{sl}_2))$ . We prepare two more lemmas for the case of  $U'_q(L(\mathfrak{sl}_2))$  to make the point clearer. Set  $(\varepsilon, \varepsilon^*) = (1, 0)$  and let  $\mathcal{T}_q$  be the augmented TD-algebra defined by  $(\text{TD})_0, (\text{TD})_1$  in (2), (3). For  $s \in \mathbb{C}^\times$ , let  $\varphi_s$  be the embedding of  $\mathcal{T}_q$  into  $U'_q(L(\mathfrak{sl}_2))$  given by (5), (6).

**Lemma 3.** Let  $V_1, V_2$  be finite-dimensional irreducible  $U'_q(L(\mathfrak{sl}_2))$ -modules. If  $V_1, V_2$  are isomorphic as  $\varphi_s(\mathcal{T}_q)$ -modules for some  $s \in \mathbb{C}^\times$ , then  $V_1, V_2$  are isomorphic as  $U'_q(L(\mathfrak{sl}_2))$ -modules.

*Proof.* By (5),  $\varphi_s(\mathcal{T}_q)$  is generated by  $se_0^+ + s^{-1}e_1^-k_1$ ,  $e_1^+$  and  $k_i^{\pm 1}$  ( $i = 0, 1$ ). Since  $\langle e_1^\pm, k_1^{\pm 1} \rangle$  is isomorphic to the quantum algebra  $U_q(\mathfrak{sl}_2)$ , the action of  $e_1^-$  on  $V_i$  ( $i = 1, 2$ ) is uniquely determined by those of  $e_1^+$ ,  $k_1^{\pm 1} \in \varphi_s(\mathcal{T}_q)$  by Lemma 2. Apparently the action of  $e_0^+$  on  $V_i$  ( $i = 1, 2$ ) is uniquely determined by those of  $se_0^+ + s^{-1}e_1^-k_1$ ,  $e_1^-$ ,  $k_1$ , and hence by that of  $\varphi_s(\mathcal{T}_q)$ . So the action of  $U'_q(L(\mathfrak{sl}_2))$  on  $V_i$  ( $i = 1, 2$ ) is uniquely determined by that of  $\varphi_s(\mathcal{T}_q)$ .  $\square$

**Lemma 4.** Let  $V$  be a finite-dimensional irreducible  $U'_q(L(\mathfrak{sl}_2))$ -module of type  $(1, 1)$ . Then there exists a finite set  $\Lambda$  of nonzero scalars such that  $V$  is irreducible as a  $\varphi_s(\mathcal{T}_q)$ -module for each  $s \in \mathbb{C}^\times - \Lambda$ .

*Proof.* For  $s \in \mathbb{C}^\times$ , regard  $V$  be a  $\varphi_s(\mathcal{T}_q)$ -module. Let  $W$  be a minimal  $\varphi_s(\mathcal{T}_q)$ -submodule of  $V$ . It is enough to show that  $W = V$  holds if  $s$  avoids finitely many scalars. By (10) with  $s_0 = 1$ , the eigenspace decomposition of  $k_1 = k_0^{-1}$  on  $V$  is  $V = \bigoplus_{i=0}^d U_{d-i}$ ,  $k_1|_{U_{d-i}} = q^{2i-d}$  ( $0 \leq i \leq d$ ). The subalgebra  $\langle e_1^\pm, k_1^{\pm 1} \rangle$  of  $U'_q(L(\mathfrak{sl}_2))$  is isomorphic to the quantum algebra  $\mathcal{U} = U_q(\mathfrak{sl}_2)$  in (15) via the correspondence of  $e_1^\pm, k_1^{\pm 1}$  to  $X^\pm, K^{\pm 1}$ . The element  $(e_1^+)^{d-2i}$  maps  $U_{d-i}$  onto  $U_i$  bijectively ( $0 \leq i \leq [d/2]$ ). Also  $(e_1^- k_1)^{d-2i}$  maps  $U_i$  onto  $U_{d-i}$  bijectively ( $0 \leq i \leq [d/2]$ ).

The element  $(e_1^+)^{d-2i}$  belongs to  $\varphi_s(\mathcal{T}_q)$ . So  $(e_1^+)^{d-2i}W \subseteq W$ . Since the mapping  $(e_1^+)^{d-2i} : U_{d-i} \rightarrow U_i$  is a bijection, we have  $\dim(W \cap U_{d-i}) \leq \dim(W \cap U_i)$  ( $0 \leq i \leq [d/2]$ ).

The element  $(e_1^- k_1)^{d-2i}$  does not belong to  $\varphi_s(\mathcal{T}_q)$ , but  $(e_0^+ + s^{-2}e_1^- k_1)^{d-2i}$  does. By (9),  $(e_0^+ + s^{-2}e_1^- k_1)^{d-2i}$  maps  $U_i$  to  $U_{d-i}$  ( $0 \leq i \leq [d/2]$ ). We want to show it is a bijection if  $s$  avoids finitely many scalars. Identify  $U_{d-i}$  with  $U_i$  as vector spaces by the bijection  $(e_1^+)^{d-2i}$  between them. Then it makes sense to consider the determinant of a linear map from  $U_i$  to  $U_{d-i}$ . Set  $t = s^{-2}$  and expand  $(e_0^+ + te_1^- k_1)^{d-2i}$  as

$$t^{d-2i}(e_1^- k_1)^{d-2i} + \text{lower terms in } t.$$

Each term of the expansion gives a linear map from  $U_i$  to  $U_{d-i}$ . So the determinant of  $(e_0^+ + te_1^- k_1)^{d-2i}|_{U_i}$  equals

$$t^{(d-2i)\dim U_i} \det(e_1^- k_1)^{d-2i}|_{U_i} + \text{lower terms in } t, \quad (16)$$

and this is a polynomial in  $t$  of degree  $(d-2i)\dim U_i$ , since  $\det(e_1^- k_1)^{d-2i}|_{U_i} \neq 0$ . Let  $\Lambda_i$  be the set of nonzero  $s$  such that  $t = s^{-2}$  is not a root of the polynomial in (16). Then if  $s \in \mathbb{C}^\times - \Lambda_i$ ,  $(e_0^+ + s^{-2}e_1^- k_1)^{d-2i}$  maps  $U_i$  to  $U_{d-i}$  bijectively.

Set  $\Lambda = \bigcup_{i=0}^{[d/2]} \Lambda_i$ . Choose  $s \in \mathbb{C}^\times - \Lambda$ . Then the mapping  $(e_0^+ + s^{-2}e_1^- k_1)^{d-2i} : U_i \rightarrow U_{d-i}$  is a bijection for  $0 \leq i \leq [d/2]$ . Since  $e_0^+ + s^{-2}e_1^- k_1$  belongs to  $\varphi_s(\mathcal{T}_q)$ , we have  $(e_0^+ + s^{-2}e_1^- k_1)^{d-2i}W \subseteq W$  and so  $\dim(W \cap U_i) \leq \dim(W \cap U_{d-i})$ . Since we have already shown  $\dim(W \cap U_{d-i}) \leq \dim(W \cap U_i)$ , we obtain  $\dim(W \cap U_i) = \dim(W \cap U_{d-i})$  ( $0 \leq i \leq [d/2]$ ). Therefore by Lemma 1, we have  $e_1^- W \subseteq W$ . Since  $(e_0^+ + s^{-2}e_1^- k_1)W \subseteq W$ , the inclusion

$e_0^+ W \subseteq W$  follows from  $e_1^- W \subseteq W$  and so  $W$  is  $U'_q(L(\mathfrak{sl}_2))$ -invariant. Thus  $W = V$  holds by the irreducibility of  $V$  as a  $U'_q(L(\mathfrak{sl}_2))$ -module.  $\square$

*Proof of Theorem 1.* We use the classification of finite-dimensional irreducible  $\mathcal{T}_q$ -modules in the case of  $(\varepsilon, \varepsilon^*) = (1, 0)$  [4, Theorem 1.18]:

- (i) A tensor product  $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$  of evaluation modules is irreducible as a  $\varphi_s(\mathcal{T}_q)$ -module if and only if  $-s^{-2} \notin S(\ell_i, a_i)$  for all  $i \in \{1, \dots, n\}$  and  $S(\ell_i, a_i), S(\ell_j, a_j)$  are in general position for all  $i, j \in \{1, \dots, n\}$ .
- (ii) Consider two tensor products  $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ ,  $V' = V(\ell') \otimes V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_m, a'_m)$  of evaluation modules and assume that they are both irreducible as a  $\varphi_s(\mathcal{T}_q)$ -module. Then  $V, V'$  are isomorphic as  $\varphi_s(\mathcal{T}_q)$ -modules if and only if  $\ell = \ell', n = m$  and  $(\ell_i, a_i) = (\ell'_i, a'_i)$  for all  $i \in \{1, \dots, n\}$  with a suitable reordering of the evaluation modules  $V(\ell_1, a_1), \dots, V(\ell_n, a_n)$ .
- (iii) Every finite-dimensional irreducible  $\mathcal{T}_q$ -module  $V$  ( $\dim V \geq 2$ ) is isomorphic to a  $\mathcal{T}_q$ -module  $V' = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$  on which  $\mathcal{T}_q$  acts via some embedding  $\varphi_s : \mathcal{T}_q \longrightarrow U'_q(L(\mathfrak{sl}_2))$ .

Part (i) of Theorem 1 follows immediately from the part (i) above, due to Lemma 4. Part (ii) of Theorem 2 follows immediately from the part (ii) above, the ‘if’ part due to Lemma 3 (and Lemma 4) and the ‘only if’ part due to Lemma 4.

We want to show part (iii) of Theorem 1. Let  $V$  be a finite-dimensional irreducible  $U'_q(L(\mathfrak{sl}_2))$ -module of type  $(1, 1)$ . By Lemma 4, there exists a nonzero scalar  $s$  such that  $V$  is irreducible as a  $\varphi_s(\mathcal{T}_q)$ -module. By the part (iii) above, the  $\mathcal{T}_q$ -module  $V$  via  $\varphi_s$  is isomorphic to some  $\mathcal{T}_q$ -module  $V' = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$  via some embedding  $\varphi_{s'}$  of  $\mathcal{T}_q$  into  $U'_q(L(\mathfrak{sl}_2))$ . Since  $k_0$  has the same eigenvalues on  $V, V'$ , we have  $s = s'$  and so  $V, V'$  are isomorphic as  $\varphi_s(\mathcal{T}_q)$ -modules. By Lemma 3,  $V, V'$  are isomorphic as  $U'_q(L(\mathfrak{sl}_2))$ -modules. This completes the proof of Theorem 1.  $\square$

### 3 Intertwiners

In this section, we show that for  $\ell, m \in \mathbb{Z}_{>0}, a \in \mathbb{C}^\times$ , there exists an intertwiner between the  $U'_q(L(\mathfrak{sl}_2))$ -modules  $V(\ell, a) \otimes V(m), V(m) \otimes V(\ell, a)$ , i.e.,

a nonzero linear map  $R$  from  $V(\ell, a) \otimes V(m)$  to  $V(m) \otimes V(\ell, a)$  such that

$$R \Delta(\xi) = \Delta(\xi) R \quad (\forall \xi \in U'_q(L(\mathfrak{sl}_2))). \quad (17)$$

If such an intertwiner  $R$  exists, then it is routinely concluded that  $V(\ell, a) \otimes V(m)$  is isomorphic to  $V(m) \otimes V(\ell, a)$  as  $U'_q(L(\mathfrak{sl}_2))$ -modules and any other intertwiner is a scalar multiple of  $R$ , since  $V(m) \otimes V(\ell, a)$  is irreducible as a  $U'_q(L(\mathfrak{sl}_2))$ -module by Theorem 1.

Using the theory of Drinfel'd polynomials [4] for the augmented TD-algebra  $\mathcal{T}_q = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)}$  with  $(\varepsilon, \varepsilon^*) = (1, 0)$ , we shall firstly show that  $V(\ell, a) \otimes V(m)$  is isomorphic to  $V(m) \otimes V(\ell, a)$  as  $U'_q(L(\mathfrak{sl}_2))$ -modules. We shall then construct an intertwiner explicitly.

Let us denote the  $U'_q(L(\mathfrak{sl}_2))$ -modules  $V(\ell, a) \otimes V(m)$ ,  $V(m) \otimes V(\ell, a)$  by  $V, V'$ :

$$V = V(\ell, a) \otimes V(m), \quad V' = V(m) \otimes V(\ell, a).$$

Recall we assume that the integers  $\ell, m$  and the scalar  $a$  are nonzero. Let us denote a standard basis of the  $U'_q(L(\mathfrak{sl}_2))$ -module  $V(\ell, a)$  (resp.  $V(m)$ ) by  $v_0, v_1, \dots, v_\ell$  (resp.  $v'_0, v'_1, \dots, v'_m$ ) in the sense of (11). Recall  $V(m)$  is an abbreviation of  $V(m, 0)$  and the action of  $U'_q(L(\mathfrak{sl}_2))$  on  $V, V'$  are via the coproduct  $\Delta$  of (12).

Let  $\mathcal{U}$  denote the subalgebra of  $U'_q(L(\mathfrak{sl}_2))$  generated by  $e_1^\pm, K_1^\pm$ . The subalgebra  $\mathcal{U}$  is isomorphic to the quantum algebra  $U_q(\mathfrak{sl}_2)$ . Let  $V(n)$  denote the irreducible  $\mathcal{U}$ -module of dimension  $n + 1$ :  $V(n)$  has a standard basis  $x_0, x_1, \dots, x_n$  on which  $\mathcal{U}$  acts as

$$\begin{cases} k_1 x_i = q^{n-2i} x_i, \\ e_1^+ x_i = [n - i + 1] x_{i-1}, \\ e_1^- x_i = [i + 1] x_{i+1}, \end{cases} \quad (18)$$

where  $[t] = [t]_q = (q^t - q^{-t})/(q - q^{-1})$ ,  $x_{-1} = x_{n+1} = 0$ . We call  $x_n$  (resp.  $x_0$ ) the *lowest (highest) weight vector*:  $k_1 x_n = q^{-n} x_n$ ,  $e_1^- x_n = 0$  ( $k_1 x_0 = q^n x_0$ ,  $e_1^+ x_0 = 0$ ). Note that  $V(\ell, a)$  is isomorphic to  $V(\ell)$  as  $\mathcal{U}$ -modules.

By the Clebsch-Gordan formula,  $V = V(\ell, a) \otimes V(m)$  is decomposed into the direct sum of  $\mathcal{U}$ -submodules  $\tilde{V}(n)$  ( $|\ell - m| \leq n \leq \ell + m$ ,  $n \equiv \ell + m \pmod{2}$ ), where  $\tilde{V}(n)$  is the unique irreducible  $\mathcal{U}$ -submodule of  $V$  isomorphic to  $V(n)$ . With  $n = \ell + m - 2\nu$ , we have

$$V = V(\ell, a) \otimes V(m) = \bigoplus_{\nu=0}^{\min\{\ell, m\}} \tilde{V}(\ell + m - 2\nu). \quad (19)$$

Let  $\tilde{x}_n$  denote a lowest weight vector of the  $\mathcal{U}$ -module  $\tilde{V}(n)$ . So

$$\begin{cases} \Delta(k_1)\tilde{x}_n = q^{-n}\tilde{x}_n, \\ \Delta(e_1^-)\tilde{x}_n = 0. \end{cases} \quad (20)$$

Since  $V$  has a basis  $\{v_{\ell-i} \otimes v'_{m-j} \mid 0 \leq i \leq \ell, 0 \leq j \leq m\}$  and  $k_1$  acts on it by  $\Delta(k_1)(v_{\ell-i} \otimes v'_{m-j}) = q^{-(\ell+m)+2(i+j)}v_{\ell-i} \otimes v'_{m-j}$ , the lowest weight vector  $\tilde{x}_n$  of  $\tilde{V}(n)$  can be expressed as

$$\tilde{x}_n = \sum_{i+j=\nu} c_j v_{\ell-i} \otimes v'_{m-j} \quad (n = \ell + m - 2\nu). \quad (21)$$

Solving  $\Delta(e_1^-)\tilde{x}_n = 0$  for the coefficients  $c_j$ , we obtain

$$\frac{c_j}{c_{j-1}} = -q^{m-2j+2} \frac{[\ell - \nu + j]}{[m - j + 1]}$$

and so with a suitable choice of  $c_0$

$$\tilde{x}_n = \sum_{j=0}^{\nu} (-1)^j q^{j(m-j+1)} [\ell - \nu + j]! [m - j]! v_{\ell-\nu+j} \otimes v'_{m-j}, \quad (22)$$

where  $n = \ell + m - 2\nu$  and  $[t]! = [t][t-1] \cdots [1]$ .

**Lemma 5.**  $\Delta(e_0^+)\tilde{x}_n = aq\tilde{x}_{n+2}$ .

*Proof.* By (12), we have  $\Delta(e_0^+) = e_0^+ \otimes 1 + k_0 \otimes e_0^+$ . By (11), the element  $e_0^+$  vanishes on  $V(m)$  and acts on  $V(\ell, a)$  as  $aqe_1^-$ . Since  $e_1^- v_{\ell-\nu+j} = [\ell - (\nu - 1) + j]v_{\ell-(\nu-1)+j}$ , the result follows from (22), using  $v_{\ell+1} = 0$ .  $\square$

**Corollary 1.** Any nonzero  $U'_q(L(\mathfrak{sl}_2))$ -submodule of  $V(\ell, a) \otimes V(m)$  contains  $\tilde{x}_{\ell+m}$ , the lowest weight vector of the  $\mathcal{U}$ -module  $V(\ell, a) \otimes V(m)$ .

We are ready to prove our second main result.

**Theorem 2.** The  $U'_q(L(\mathfrak{sl}_2))$ -modules  $V(\ell, a) \otimes V(m)$ ,  $V(m) \otimes V(\ell, a)$  are isomorphic for every  $\ell, m \in \mathbb{Z}_{>0}$ ,  $a \in \mathbb{C}^\times$ .

*Proof.* Let  $\mathcal{T}_q = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)}$  be the augmented TD-algebra with  $(\varepsilon, \varepsilon^*) = (1, 0)$ . Let  $\varphi_s : \mathcal{T}_q \longrightarrow U'_q(L(\mathfrak{sl}_2))$  denote the embedding of  $\mathcal{T}_q$  into  $U'_q(L(\mathfrak{sl}_2))$  given

in (6). By Theorem 5.2 of [4], the Drinfel'd polynomial  $P_V(\lambda)$  of the  $\varphi_s(\mathcal{T}_q)$ -module  $V = V(\ell, a) \otimes V(m)$  turns out to be

$$P_V(\lambda) = \lambda^m \prod_{i=0}^{\ell-1} (\lambda + aq^{2i-\ell+1}). \quad (23)$$

(Note that the parameter  $s$  of the embedding  $\varphi_s$  does not appear in  $P_V(\lambda)$ . So the polynomial  $P_V(\lambda)$  can be called the Drinfel'd polynomial attached to the  $U'_q(L(\mathfrak{sl}_2))$ -module  $V$ .)

Let  $W$  be a minimal  $U'_q(L(\mathfrak{sl}_2))$ -submodule of  $V$ . By Corollary 1,  $W$  contains the lowest and hence highest weight vectors of  $V$ . In particular, the irreducible  $U'_q(L(\mathfrak{sl}_2))$ -module  $W$  is of type  $(1, 1)$ . By Lemma 4, there exists a finite set  $\Lambda$  of nonzero scalars such that  $W$  is irreducible as a  $\varphi_s(\mathcal{T}_q)$ -module for any  $s \in \mathbb{C}^\times - \Lambda$ . By the definition [4, (25)], the Drinfel'd polynomial  $P_W(\lambda)$  of the irreducible  $\varphi_s(\mathcal{T}_q)$ -module  $W$  coincides with  $P_V(\lambda)$ :

$$P_W(\lambda) = P_V(\lambda). \quad (24)$$

By Theorem 1,  $V' = V(m) \otimes V(\ell, a)$  is irreducible as a  $U'_q(L(\mathfrak{sl}_2))$ -module. So by Lemma 4, there exists a finite set  $\Lambda'$  of nonzero scalars such that  $V'$  is irreducible as a  $\varphi_s(\mathcal{T}_q)$ -module for any  $s \in \mathbb{C}^\times - \Lambda'$ . By Theorem 5.2 of [4], the Drinfel'd polynomial  $P_{V'}(\lambda)$  of the irreducible  $\varphi_s(\mathcal{T}_q)$ -module  $V'$  coincides with  $P_V(\lambda)$  :

$$P_{V'}(\lambda) = P_V(\lambda). \quad (25)$$

Both of the irreducible  $\varphi_s(\mathcal{T}_q)$ -modules  $W, V'$  have type  $s$ , diameter  $d = \ell + m$  and the Drinfel'd polynomial  $P_V(\lambda)$ . By Theorem 1.9' of [4],  $W$  and  $V'$  are isomorphic as  $\varphi_s(\mathcal{T}_q)$ -modules. By Lemma 3,  $W$  and  $V'$  are isomorphic as  $U'_q(L(\mathfrak{sl}_2))$ -modules. In particular,  $\dim W = \dim V'$ . Since  $\dim V' = \dim V$ , we have  $W = V$ , i.e.,  $V$  and  $V'$  are isomorphic as  $U'_q(L(\mathfrak{sl}_2))$ -modules.  $\square$

Finally we want to construct an intertwiner  $R$  between the irreducible  $U'_q(L(\mathfrak{sl}_2))$ -modules  $V, V'$ . Regard  $V' = V(m) \otimes V(\ell, a)$  as a  $\mathcal{U}$ -module. By the Clebsch-Gordan formula, we have the direct sum decomposition

$$V' = V(m) \otimes V(\ell, a) = \bigoplus_{\nu=0}^{\min\{\ell, m\}} \tilde{V}'(\ell + m - 2\nu), \quad (26)$$

where  $\tilde{V}'(n)$  is the unique irreducible  $\mathcal{U}$ -submodule of  $V'$  isomorphic to  $V(n)$  ( $n = \ell + m - 2\nu$ ). Let  $\tilde{x}'_n$  be a lowest weight vector of the  $\mathcal{U}$ -module  $\tilde{V}'(n)$ . By (22), we have

$$\tilde{x}'_n = \sum_{j=0}^{\nu} (-1)^j q^{j(\ell-j+1)} [m - \nu + j]! [\ell - j]! v'_{m-\nu+j} \otimes v_{\ell-j} \quad (27)$$

up to a scalar multiple, where  $n = \ell + m - 2\nu$ . It can be easily checked as in Lemma 5 that the lowest weight vectors  $\tilde{x}'_n$  ( $n = \ell + m - 2\nu$ ,  $0 \leq \nu \leq \min\{\ell, m\}$ ) are related by

$$(e_1^- \otimes 1) \tilde{x}'_n = \tilde{x}'_{n+2}, \quad (28)$$

where  $V' = V(m) \otimes V(\ell, a)$  is regarded as a  $(\mathcal{U} \otimes \mathcal{U})$ -module in the natural way.

**Lemma 6.**  $\Delta(e_0^+) \tilde{x}'_n = -aq \cdot q^{n+2} \tilde{x}'_{n+2}$ .

*Proof.* We have  $\Delta(e_0^+) \tilde{x}'_n = aq (k_1^{-1} \otimes e_1^-) \tilde{x}'_n$ , since  $\Delta(e_0^+) = e_0^+ \otimes 1 + k_0 \otimes e_0^+$ , and  $e_0^+$  vanishes on  $V(m)$  and acts on  $V(\ell, a)$  as  $aq e_1^-$ . Express  $k_1^{-1} \otimes e_1^-$  as  $k_1^{-1} \otimes e_1^- = (k_1^{-1} \otimes 1)(1 \otimes e_1^-) = (k_1^{-1} \otimes 1)(\Delta(e_1^-) - e_1^- \otimes k_1^{-1}) = (k_1^{-1} \otimes 1)\Delta(e_1^-) - k_1^{-1} e_1^- \otimes k_1^{-1} = (k_1^{-1} \otimes 1)\Delta(e_1^-) - q^2(e_1^- \otimes 1)\Delta(k_1^{-1})$ . Since  $\Delta(e_1^-) \tilde{x}'_n = 0$ ,  $\Delta(k_1^{-1}) \tilde{x}'_n = q^n \tilde{x}'_n$ , the result follows from (28).  $\square$

There exists a unique linear map

$$R_n : V = V(\ell, a) \otimes V(m) \longrightarrow \tilde{V}'(n)$$

that commutes with the action of  $\mathcal{U}$  and sends  $\tilde{x}_n$  to  $\tilde{x}'_n$ . The linear map  $R_n$  vanishes on  $\tilde{V}(t)$  for  $t \neq n$  and affords an isomorphism between  $\tilde{V}(n)$  and  $\tilde{V}'(n)$  as  $\mathcal{U}$ -modules. If  $R$  is an intertwiner in the sense of (17),  $R$  can be expressed as

$$R = \sum_{\nu=0}^{\min\{\ell, m\}} \alpha_\nu R_{\ell+m-2\nu}, \quad (29)$$

regarding  $R$  as an intertwiner for the  $\mathcal{U}$ -modules  $V, V'$ . By (17), we have

$$R \Delta(e_0^+) = \Delta(e_0^+) R. \quad (30)$$



Apply (30) to the lowest weight vector  $\tilde{x}_n$  in (22). By Lemma 5,  $\Delta(e_0^+) \tilde{x}_n = aq \tilde{x}_{n+2}$  and so with  $n = \ell + m - 2\nu$ , we have

$$R \Delta(e_0^+) \tilde{x}_n = aq \alpha_{\nu-1} \tilde{x}'_{n+2}. \quad (31)$$

On the other hand,  $R \tilde{x}_n = \alpha_\nu \tilde{x}'_n$  ( $n = \ell + m - 2\nu$ ) and so by Lemma 6, we have

$$\Delta(e_0^+) R \tilde{x}_n = -aq \alpha_\nu q^{n+2} \tilde{x}'_{n+2}. \quad (32)$$

By (31), (32), we have  $\alpha_\nu / \alpha_{\nu-1} = -q^{-n-2} = -q^{-\ell-m+2(\nu-1)}$  and so

$$\alpha_\nu = (-1)^\nu q^{-\nu(\ell+m-\nu+1)} \quad (33)$$

with a suitable choice of  $\alpha_0$ . An intertwiner exists by Theorem 2. If it exists, it has to be in the form of (29), (33). Thus we obtain our third main result.

**Theorem 3.** The linear map

$$R = \sum_{\nu=0}^{\min\{\ell, m\}} (-1)^\nu q^{-\nu(\ell+m-\nu+1)} R_{\ell+m-2\nu}$$

is an intertwiner between the  $U'_q(L(\mathfrak{sl}_2))$ -modules  $V(\ell, a) \otimes V(m)$ ,  $V(m) \otimes V(\ell, a)$ .

## References

- [1] G. Benkart, P. Terwilliger, Irreducible modules for the quantum affine algebra  $U_q(\widehat{\mathfrak{sl}}_2)$  and its Borel subalgebra, J. Algebra 282 (2004) 172-194.
- [2] V. Chari, A. Pressley, Quantum affine algebras, Commun. Math. Phys. 142 (1991) 261-283.
- [3] T. Ito, K. Tanabe, P. Terwilliger, Some algebra related to P- and Q-polynomial association schemes, in: Codes and Association Schemes (Piscataway NJ, 1999), Amer. Math. Soc., Providence RI, 2001, pp. 167-192.
- [4] T. Ito, P. Terwilliger, The Augmented Tridiagonal Algebra, Kyushu J. Math. 64 (2010) 81-144.

Tomoya Hattai  
Division of Mathematical and Physical Sciences  
Kanazawa University  
Kakuma-machi, Kanazawa 920–1192, Japan

Tatsuro Ito  
Division of Mathematical and Physical Sciences  
Kanazawa University  
Kakuma-machi, Kanazawa 920–1192, Japan  
E-mail: [ito@se.kanazawa-u.ac.jp](mailto:ito@se.kanazawa-u.ac.jp)